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Trajectory Optimization Using Eccentric Longitude Formulation

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The problem of optimal transfer using a set of nonsingular orbit elements where the eccentric longitude represents the sixth state variable is extended and completed by fully considering the six-state dynamics. Because the eccentric longitude appears in the right-hand sides of the dynamic equations, the use of this particular formulation removes the need for solving the transcendental Kepler equation at each integration step, thereby easing, to some extent, the numerical computations. Furthermore, because the eccentric longitude is being integrated directly, it effectively becomes an independent orbital element such that the adjoint differential equations are derived by assuming that this longitude is independent of the other elements. The variational Hamiltonian is constant throughout the transfer with the boundary conditions given simply in terms of the elements. An example of a general minimum-time transfer using continuous low-thrust is generated duplicating a previous result using the mean longitude formulation to validate the mathematical derivations.

Nomenclature

a	= semimajor axis, km
E	= eccentric anomaly
e	= eccentricity
F	= eccentric longitude, $E + \omega + \Omega$
f	= thrust magnitude, N
$\boldsymbol{f}$	= thrust vector, N
f_t	= thrust acceleration magnitude, f/m
$\hat{f}, \hat{g}, \hat{w}$	= unit vectors along axes of direct equinoctial frame
G	= $(1 - h^2 - k^2)^{1/2}$
h	= $e \sin(\omega + \Omega)$
i	= orbit inclination
K	= $1 + p^2 + q^2$
k	= $e \cos(\omega + \Omega)$
M	= mean anomaly
m	= spacecraft mass, kg
n	= orbit mean motion, $\mu^{1/2}a^{-3/2}$, rad/s
p	= $\tan(i/2) \sin \Omega$
p'	= orbit parameter, $a(1 - e^2)$, km
q	= $\tan(i/2) \cos \Omega$
r	= radial distance, km
$\dot{\boldsymbol{r}}$	= velocity vector, km/s
s_F, c_F	= $\sin F, \cos F$, etc.
$\hat{u}$	= unit vector in the direction of $\boldsymbol{f}$
λ	= mean longitude, $M + \omega + \Omega$
μ	= Earth gravity constant, km^3/s^2
Ω	= longitude of ascending node
ω	= argument of periapse

Introduction

EQUINOCTIAL orbit elements have been used[1−3] to solve optimal low-thrust transfer problems by considering only the five slowly varying orbit elements. The dynamic equations based on these elements are singularity free for both zero eccentricity and zero inclination and are valid for any elliptical orbit because they are exact nonlinear equations that describe the exact motion in a central force field and are perturbed by the thrust acceleration. In problems of low-thrust transfer, the adoption of orbital elements allows for the technique of averaging to be applied because these elements vary slowly during an entire orbit. When precision integration is used instead to obtain exact solutions, the Cartesian formulation that is also singularity free can also be used to generate optimal trajectories. However, with the boundary conditions being given in terms of the elements, it is more natural to use an element-based formulation even though the Cartesian formulation is mathematically very simple. In an element-based formulation, the equations of motion contain an accessory variable in the right-hand sides, namely the eccentric anomaly, if classical elements are considered, or eccentric longitude, if equinoctial variables are considered instead. These equations can also be written in terms of the true anomaly or eccentric anomaly, respectively, as the accessory variable.

When all six elements are considered, the differential equation for the fast element or sixth independent variable can be written as the time rate of change of the mean longitude, the eccentric longitude, or the true longitude, the accessory variable appearing in the right-hand sides of these dynamic equations being the eccentric longitude or the true longitude but not the mean longitude. The reason the mean longitude cannot be used as the accessory variable is that the position vector cannot be written explicitly in terms of the mean longitude. If we select the mean longitude as the sixth independent variable, then we must solve for the eccentric longitude through Kepler's equation at each integration step to evaluate the right-hand sides of all six differential equations. This additional effort in solving Kepler's equation is clearly not needed if the eccentric longitude is selected as the sixth variable instead. The particular choice of the sixth variable has important implications on how the differential equations for the adjoints are written.[4−12] This observation holds true even if only the first five dynamic equations for the slowly varying elements are considered and the sixth equation is neglected. There is no ambiguity if all six equations are considered provided that the dependence of the eccentric longitude on the other elements is accounted for when the mean longitude is selected as the sixth variable and is considered independent of the other elements when the eccentric longitude itself is selected as the sixth variable. The formulation for the case where the eccentric longitude is selected as the sixth state variable while considering only the differential equations for the first five slowly varying elements[1,2] is extended in this paper by considering all six differential equations for an exact description of the orbit motion. The full six-element formulation for the case where the mean longitude represents the fast variable was developed later.[7]

The next two sections show all the pertinent derivations with a duplication of the general example of Ref. 7 to validate the mathematics of this version.

Equations of Motion in Terms of the Eccentric Longitude

The equations of motion in terms of the eccentric longitude for the thrust perturbation are given by[5]

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$$\dot{a} = \left(\frac{\partial a}{\partial \dot{\boldsymbol{r}}}\right)^T \cdot \hat{u} f_t \tag{1}$$

$$\dot{h} = \left(\frac{\partial h}{\partial \dot{\boldsymbol{r}}}\right)^T \cdot \hat{u} f_t \tag{2}$$

$$\dot{k} = \left(\frac{\partial k}{\partial \dot{\boldsymbol{r}}}\right)^T \cdot \hat{u} f_t \tag{3}$$

$$\dot{p} = \left(\frac{\partial p}{\partial \dot{\boldsymbol{r}}}\right)^T \cdot \hat{u} f_t \tag{4}$$

$$\dot{q} = \left(\frac{\partial q}{\partial \dot{\boldsymbol{r}}}\right)^T \cdot \hat{u} f_t \tag{5}$$

$$\dot{F} = \frac{na}{r} + \left(\frac{\partial F}{\partial \dot{\boldsymbol{r}}}\right)^T \cdot \hat{u} f_t \tag{6}$$

The thrust vector $\boldsymbol{f} = f\hat{u}$ is measured in the direct equinoctial frame $(\hat{f}, \hat{g}, \hat{w})$ with $\hat{f}$ and $\hat{g}$ contained in the instantaneous orbit plane and with $\hat{f}$ obtained through a clockwise rotation of an angle Ω from the direction of the ascending node. The position and velocity vectors $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ are written as $\boldsymbol{r} = X_1\hat{f} + Y_1\hat{g}$ and $\dot{\boldsymbol{r}} = \dot{X}_1\hat{f} + \dot{Y}_1\hat{g}$. The various preceding components are given in terms of the equinoctial orbit elements as well as the eccentric longitude F by

$$X_1 = a\left[(1 - h^2\beta)c_F + hk\beta s_F - k\right] \tag{7}$$

$$Y_1 = a\left[hk\beta c_F + (1 - k^2\beta)s_F - h\right] \tag{8}$$

$$\dot{X}_1 = a^2 n r^{-1}\left[hk\beta c_F - (1 - h^2\beta)s_F\right] \tag{9}$$

$$\dot{Y}_1 = a^2 n r^{-1}\left[(1 - k^2\beta)c_F - hk\beta s_F\right] \tag{10}$$

Here $\beta = 1/(1 + G)$, and $r = a(1 - kc_F - hs_F)$.

The partials appearing in Eqs. (1–6) are given by

$$\frac{\partial a}{\partial \dot{\boldsymbol{r}}} = 2a^{-1}n^{-2}(\dot{X}_1\hat{f} + \dot{Y}_1\hat{g}) = M_{11}\hat{f} + M_{12}\hat{g} + M_{13}\hat{w} \tag{11}$$

$$\frac{\partial h}{\partial \dot{\boldsymbol{r}}} = Gn^{-1}a^{-2}\left[\left(\frac{\partial X_1}{\partial k} - h\beta\frac{\dot{X}_1}{n}\right)\hat{f} + \left(\frac{\partial Y_1}{\partial k} - h\beta\frac{\dot{Y}_1}{n}\right)\hat{g}\right] + k(qY_1 - pX_1)n^{-1}a^{-2}G^{-1}\hat{w} = M_{21}\hat{f} + M_{22}\hat{g} + M_{23}\hat{w} \tag{12}$$

$$\frac{\partial k}{\partial \dot{\boldsymbol{r}}} = -Gn^{-1}a^{-2}\left[\left(\frac{\partial X_1}{\partial h} + k\beta\frac{\dot{X}_1}{n}\right)\hat{f} + \left(\frac{\partial Y_1}{\partial h} + k\beta\frac{\dot{Y}_1}{n}\right)\hat{g}\right] - h(qY_1 - pX_1)n^{-1}a^{-2}G^{-1}\hat{w} = M_{31}\hat{f} + M_{32}\hat{g} + M_{33}\hat{w} \tag{13}$$

$$\frac{\partial p}{\partial \dot{\boldsymbol{r}}} = KY_1\frac{n^{-1}a^{-2}G^{-1}}{2}\hat{w} = M_{41}\hat{f} + M_{42}\hat{g} + M_{43}\hat{w} \tag{14}$$

$$\frac{\partial p}{\partial \dot{\boldsymbol{r}}} = KX_1\frac{n^{-1}a^{-2}G^{-1}}{2}\hat{w} = M_{51}\hat{f} + M_{52}\hat{g} + M_{53}\hat{w} \tag{15}$$

$$\frac{\partial F}{\partial \dot{\boldsymbol{r}}} = \frac{a}{r}\left[\frac{\partial \lambda}{\partial \dot{\boldsymbol{r}}} + s_F\frac{\partial k}{\partial \dot{\boldsymbol{r}}} - c_F\frac{\partial h}{\partial \dot{\boldsymbol{r}}}\right] = M^F_{61}\hat{f} + M^F_{62}\hat{g} + M^F_{63}\hat{w} \tag{16}$$

and where

$$\frac{\partial \lambda}{\partial \dot{\boldsymbol{r}}} = n^{-1}a^{-2}\left[-2X_1 + G\left(h\beta\frac{\partial X_1}{\partial h} + k\beta\frac{\partial X_1}{\partial k}\right)\right]\hat{f} + n^{-1}a^{-2}\left[-2Y_1 + G\left(h\beta\frac{\partial Y_1}{\partial h} + k\beta\frac{\partial Y_1}{\partial k}\right)\right]\hat{g} + n^{-1}a^{-2}G^{-1}(qY_1 - pX_1)\hat{w} = M_{61}\hat{f} + M_{62}\hat{g} + M_{63}\hat{w} \tag{17}$$

with

$$\frac{\partial X_1}{\partial h} = a\left[-(hc_F - ks_F)\left(\beta + \frac{h^2\beta^3}{1 - \beta}\right) - \frac{a}{r}c_F(h\beta - s_F)\right] \tag{18}$$

$$\frac{\partial X_1}{\partial k} = -a\left[(hc_F - ks_F)\frac{hk\beta^3}{1 - \beta} + 1 + \frac{a}{r}s_F(s_F - h\beta)\right] \tag{19}$$

$$\frac{\partial Y_1}{\partial h} = a\left[(hc_F - ks_F)\frac{hk\beta^3}{1 - \beta} - 1 + \frac{a}{r}c_F(k\beta - c_F)\right] \tag{20}$$

$$\frac{\partial Y_1}{\partial k} = a\left[(hc_F - ks_F)\left(\beta + \frac{k^2\beta^3}{1 - \beta}\right) + \frac{a}{r}s_F(c_F - k\beta)\right] \tag{21}$$

The derivations that lead to these equations are given in great detail in Refs. 4 and 5. The partial $\partial F/\partial \dot{\boldsymbol{r}}$ is written directly in terms of the partials $\partial\lambda/\partial\dot{\boldsymbol{r}}$, $\partial h/\partial\dot{\boldsymbol{r}}$, and $\partial k/\partial\dot{\boldsymbol{r}}$. Because the partials of matrix M with respect to the elements were developed[7] using λ as the fast element, it is straightforward to use these partials with minor modifications to generate the partials of $\partial F/\partial\dot{\boldsymbol{r}}$ with respect to the elements needed in the Euler–Lagrange equations. It is for this reason that $\partial F/\partial\dot{\boldsymbol{r}}$ is not written explicitly and is left as in Eq. (16). The explicit form of $\partial F/\partial\dot{\boldsymbol{r}}$ is written as

$$\frac{\partial F}{\partial \dot{\boldsymbol{r}}} = \frac{1}{nar}\left\{\left[-2X_1 + G(h\beta - s_F)\frac{\partial X_1}{\partial h} + G(k\beta - c_F)\frac{\partial X_1}{\partial k} - \beta G(ks_F - hc_F)\frac{\dot{X}_1}{n}\right]\hat{f} + \left[-2Y_1 + G(h\beta - s_F)\frac{\partial Y_1}{\partial h} + G(k\beta - c_F)\frac{\partial Y_1}{\partial k} - \beta G(ks_F - hc_F)\frac{\dot{Y}_1}{n}\right]\hat{g} + \frac{r}{aG}(qY_1 - pX_1)\hat{w}\right\}$$

The basic equations of motion can be written in a compact form as

$$\dot{z} = M\hat{u}f_t + (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad na/r)^T \tag{22}$$

Here $z = (a \ \ h \ \ k \ \ q \ \ p \ \ F)^T$ is the state vector and the 6×3 matrix M is given by

$$M = \begin{bmatrix} \left(\frac{\partial a}{\partial \dot{\boldsymbol{r}}}\right)^T \\ \left(\frac{\partial h}{\partial \dot{\boldsymbol{r}}}\right)^T \\ \left(\frac{\partial k}{\partial \dot{\boldsymbol{r}}}\right)^T \\ \left(\frac{\partial p}{\partial \dot{\boldsymbol{r}}}\right)^T \\ \left(\frac{\partial q}{\partial \dot{\boldsymbol{r}}}\right)^T \\ \left(\frac{\partial F}{\partial \dot{\boldsymbol{r}}}\right)^T \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \\ M_{41} & M_{42} & M_{43} \\ M_{51} & M_{52} & M_{53} \\ M^F_{61} & M^F_{62} & M^F_{63} \end{bmatrix}$$

These are the variation of parameter equations for this particular set of state variables, namely, a, h, k, p, q, F, and they are obtained directly from the corresponding equations of the fundamental set a, h, k, p, q, λ_0, where λ_0 represents the mean longitude at epoch. The matrix of the Poisson brackets of the fundamental equinoctial set is obtained by applying the following transformation equation to the matrix of the Poisson brackets of the fundamental classical set $a, e, i, \Omega, \omega, M_0$:

$$[p_{\boldsymbol\alpha}, p_\beta] = \left[\frac{\partial p_{\boldsymbol\alpha}}{\partial a_\lambda}\right][(a_\lambda, a_\mu)]\left[\frac{\partial p_\beta}{\partial a_\mu}\right]^T$$

where

$$[(a_\lambda, a_\mu)] = P_{\text{cl}}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -2\left(\frac{a}{\mu}\right)^{\frac12} \\ & 0 & 0 & 0 & \frac{(1-e^2)^{\frac12}}{e(\mu a)^{\frac12}} & \frac{-(1-e^2)}{e(\mu a)^{\frac12}} \\ & & 0 & \frac{1}{(\mu p')^{\frac12} s_i} & \frac{-c_i}{(\mu p')^{\frac12} s_i} & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ -\text{Sym} & & & & & 0 \end{bmatrix}$$

$$\left[\frac{\partial p_{\boldsymbol\alpha}}{\partial a_\lambda}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & s_{\omega+\Omega} & 0 & ec_{\omega+\Omega} & ec_{\omega+\Omega} & 0 \\ 0 & c_{\omega+\Omega} & 0 & -es_{\omega+\Omega} & -es_{\omega+\Omega} & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & \frac{s_\Omega}{2\cos^2(i/2)} & \tan\frac{i}{2}c_\Omega & 0 & 0 \\ 0 & 0 & \frac{c_\Omega}{2\cos^2(i/2)} & -\tan\frac{i}{2}s_\Omega & 0 & 0 \end{bmatrix}$$

Converting the elements of the preceding two matrices from the classical elements to the equinoctial elements and carrying out the transformation yields

$$[(p_{\boldsymbol\alpha}, p_\beta)]$$

$$= \begin{bmatrix} (a,a) & (a,h) & (a,k) & (a,\lambda_0) & (a,p) & (a,q) \\ (h,a) & (h,h) & (h,k) & (h,\lambda_0) & (h,p) & (h,q) \\ (k,a) & (k,h) & (k,k) & (k,\lambda_0) & (k,p) & (k,q) \\ (\lambda_0,a) & (\lambda_0,h) & (\lambda_0,k) & (\lambda_0,\lambda_0) & (\lambda_0,p) & (\lambda_0,q) \\ (p,a) & (p,h) & (p,k) & (p,\lambda_0) & (p,p) & (p,q) \\ (q,a) & (q,h) & (q,k) & (q,\lambda_0) & (q,p) & (q,q) \end{bmatrix}$$

$$= \frac{1}{na^2}\begin{bmatrix} 0 & 0 & 0 & -2a & 0 & 0 \\ & 0 & -G & h\beta G & -kpK/2G & -kqk/2G \\ & & 0 & k\beta G & hpK/2G & hqK/2G \\ & & & 0 & -pK/2G & -qK/2G \\ -\text{Sym} & & & & 0 & -K^2/4G \\ & & & & & 0 \end{bmatrix}$$

The partials of the equinoctial elements of the fundamental set with respect to the velocity vector $\dot{\boldsymbol r}$ are obtained from

$$\frac{\partial p_{\boldsymbol\alpha}}{\partial \dot{\boldsymbol r}} = -\sum_{\beta=1}^{6}(p_{\boldsymbol\alpha}, p_\beta)\frac{\partial \boldsymbol r}{\partial p_\beta}$$

such that for λ_0

$$\frac{\partial\lambda_0}{\partial\dot{\boldsymbol r}} = -(\lambda_0, a)\frac{\partial\boldsymbol r}{\partial a} - (\lambda_0, h)\frac{\partial\boldsymbol r}{\partial h} - (\lambda_0, k)\frac{\partial\boldsymbol r}{\partial k}$$

$$-(\lambda_0, p)\frac{\partial\boldsymbol r}{\partial p} - (\lambda_0, q)\frac{\partial\boldsymbol r}{\partial q} - (\lambda_0, \lambda_0)\frac{\partial\boldsymbol r}{\partial\lambda_0}$$

The partials of $\boldsymbol r$ with respect to the elements are generated by allowing for the variation of F with respect to the elements, with

$$\frac{\partial F}{\partial a} = -\frac{3n}{2r}t, \qquad \frac{\partial F}{\partial h} = -\frac{a}{r}c_F, \qquad \frac{\partial F}{\partial k} = \frac{a}{r}s_F, \qquad \frac{\partial F}{\partial\lambda} = \frac{a}{r}$$

with the latter partial used in

$$\frac{\partial\boldsymbol r}{\partial\lambda} = \frac{\partial\boldsymbol r}{\partial F}\frac{\partial F}{\partial\lambda}$$

such that

$$\frac{\partial\boldsymbol r}{\partial\lambda_0} = \frac{\partial\boldsymbol r}{\partial\lambda}\frac{\partial\lambda}{\partial\lambda_0}$$

with $\partial\lambda/\partial\lambda_0 = 1$. These relevant partials of $\boldsymbol r$ with respect to the elements are shown[4,11] with all the details of their derivations. They are

$$\frac{\partial\boldsymbol r}{\partial a} = \left(\frac{X_1}{a} - \frac{3t}{2a}\dot X_1\right)\hat f + \left(\frac{Y_1}{a} - \frac{3t}{2a}\dot Y_1\right)\hat g$$

$$\frac{\partial\boldsymbol r}{\partial h} = \frac{\partial X_1}{\partial h}\hat f + \frac{\partial Y_1}{\partial h}\hat g$$

$$\frac{\partial\boldsymbol r}{\partial k} = \frac{\partial X_1}{\partial k}\hat f + \frac{\partial Y_1}{\partial k}\hat g$$

$$\frac{\partial\boldsymbol r}{\partial p} = \frac{2}{K}[q(Y_1\hat f - X_1\hat g) - X_1\hat w]$$

$$\frac{\partial\boldsymbol r}{\partial q} = \frac{2}{K}[p(X_1\hat g - Y_1\hat f) + Y_1\hat w]$$

$$\frac{\partial\boldsymbol r}{\partial\lambda_0} = \frac{\dot X_1}{n}\hat f + \frac{\dot Y_1}{n}\hat g$$

Thus

$$\frac{\partial\lambda_0}{\partial\dot{\boldsymbol r}} = \frac{1}{na^2}\left\{\left[-2X_1 + 3\dot X_1 t + G\beta\left(h\frac{\partial X_1}{\partial h} + k\frac{\partial X_1}{\partial k}\right)\right]\hat f\right.$$

$$\left. + \left[-2Y_1 + 3\dot Y_1 t + G\beta\left(h\frac{\partial Y_1}{\partial h} + k\frac{\partial Y_1}{\partial k}\right)\right]\hat g + \frac{1}{G}(qY_1 - pX_1)\hat w\right\}$$

The explicit appearance of time in the preceding equation is due to the selection of the epoch variable, namely the epoch mean longitude λ_0, which is eliminated by replacing the variable λ_0 by λ. From $\lambda = \lambda_0 + nt$,

$$\frac{\partial\lambda}{\partial\dot{\boldsymbol r}} = \frac{\partial\lambda_0}{\partial\dot{\boldsymbol r}} - \frac{3}{na^2}t\dot{\boldsymbol r}$$

Thus the partials of the elements with respect to the velocity vector for the set a, h, k, p, q, λ are given by Eqs. (11–15), which remain unchanged, and Eq. (17), which replaces the partial $\partial\lambda_0/\partial\dot{\boldsymbol r}$ shown earlier. The differential equation for the sixth variable is thus changed from $\dot\lambda_0 = (\partial\lambda_0/\partial\dot{\boldsymbol r})^T\hat u f_t$ to $\dot\lambda = n + (\partial\lambda/\partial\dot{\boldsymbol r})^T\hat u f_t$ when λ_0 is replaced by λ, whereas the first five equations (1–5) remain identical. Equation (6) is obtained from Kepler's equation $\lambda = F - ks_F + hc_F$, which yields $\dot\lambda = (1 - kc_F - hs_F)\dot F - s_F\dot k + c_F\dot h$ such that

$$\dot F = \frac{a}{r}\left[n + \left(\frac{\partial\lambda}{\partial\dot{\boldsymbol r}}\right)^T\hat u f_t\right] + \frac{a}{r}\left[s_F\left(\frac{\partial k}{\partial\dot{\boldsymbol r}}\right)^T - c_F\left(\frac{\partial h}{\partial\dot{\boldsymbol r}}\right)^T\right]\hat u f_t$$

which is further written as

$$\dot F = \frac{na}{r} + \left(\frac{\partial F}{\partial\dot{\boldsymbol r}}\right)^T\hat u f_t$$

because from $\lambda = F - ks_F + hc_F$ we have

$$\frac{\partial\lambda}{\partial\dot{\boldsymbol r}} = (1 - kc_F - hs_F)\frac{\partial F}{\partial\dot{\boldsymbol r}} - s_F\frac{\partial k}{\partial\dot{\boldsymbol r}} + c_F\frac{\partial h}{\partial\dot{\boldsymbol r}}$$

and therefore

$$\frac{\partial F}{\partial \dot{r}} = \frac{a}{r}\left[\frac{\partial \lambda}{\partial \dot{r}} + s_F\frac{\partial k}{\partial \dot{r}} - c_F\frac{\partial h}{\partial \dot{r}}\right]$$

The Hamiltonian corresponding to this set (a, h, k, p, q, F) can now be written in compact form as

$$H = \boldsymbol{\lambda}_z^T\dot{z} = \boldsymbol{\lambda}_z^T M(z, F) f_t\hat{u} + \lambda_F(na/r) \tag{23}$$

This Hamiltonian is maximized by making $\hat{u}$ be parallel to $\boldsymbol{\lambda}_z^T M(z, F)$. The Euler–Lagrange equations for the set (a, h, k, p, q, F) can now be written in a straightforward manner after making use of the following partials:

$$\frac{\partial r}{\partial a} = \frac{r}{a}, \qquad \frac{\partial r}{\partial h} = -as_F, \qquad \frac{\partial r}{\partial k} = -ac_F$$

$$\frac{\partial r}{\partial F} = a(ks_F - hc_F), \qquad \frac{\partial F}{\partial a} = 0, \qquad \frac{\partial F}{\partial h} = 0, \qquad \frac{\partial F}{\partial k} = 0$$

The last three partials are identically equal to zero because F is being integrated and must therefore be considered an orbital element such that it is now independent of the other in-plane elements a, h, and k. Likewise, from $r = a(1 - kc_F - hs_F)$, the partials $\partial r/\partial h$ and $\partial r/\partial k$ do not account for the variation of F with respect to h and k. This dependence had to be accounted for when the system of dynamic equations was originally generated. In taking the partial derivatives of H with respect to the six elements, the eccentric longitude F, which appears in the right-hand side of all six differential equations in the quantities X_1, Y_1, $\dot{X}_1$, $\dot{Y}_1$, $\partial X_1/\partial h$, $\partial X_1/\partial k$, $\partial Y_1/\partial h$, $\partial Y_1/\partial k$ as well as r and the trigonometric functions s_F and c_F, must be held fixed because it is being integrated directly, becoming independent of the other five state variables a, h, k, p, and q. Now the Euler–Lagrange equations for the adjoints are given by

$$\dot{\boldsymbol{\lambda}}_z = -\frac{\partial H}{\partial z} = -\boldsymbol{\lambda}_z^T\frac{\partial M}{\partial z}f_t\hat{u} - \lambda_F\frac{\partial(na/r)}{\partial z} \tag{24}$$

Explicitly, and using the summation notation with $i = 1, 2, 3$ and $u_1 = u_f$, $u_2 = u_g$, and $u_3 = u_w$, we have the following differential equations:

$$\dot{\lambda}_a = -\frac{\partial H}{\partial a} = \left[-\lambda_a\frac{\partial M_{1i}}{\partial a}u_i - \lambda_h\frac{\partial M_{2i}}{\partial a}u_i - \lambda_k\frac{\partial M_{3i}}{\partial a}u_i - \lambda_p\frac{\partial M_{4i}}{\partial a}u_i - \lambda_q\frac{\partial M_{5i}}{\partial a}u_i - \lambda_F\frac{\partial M_{6i}^F}{\partial a}u_i\right]f_t - \lambda_F\frac{\partial}{\partial a}\left(\frac{na}{r}\right) \tag{25}$$

$$\dot{\lambda}_h = -\frac{\partial H}{\partial h} = \left[-\lambda_a\frac{\partial M_{1i}}{\partial h}u_i - \lambda_h\frac{\partial M_{2i}}{\partial h}u_i - \lambda_k\frac{\partial M_{3i}}{\partial h}u_i - \lambda_p\frac{\partial M_{4i}}{\partial h}u_i - \lambda_q\frac{\partial M_{5i}}{\partial h}u_i - \lambda_F\frac{\partial M_{6i}^F}{\partial h}u_i\right]f_t - \lambda_F\frac{\partial}{\partial h}\left(\frac{na}{r}\right) \tag{26}$$

$$\dot{\lambda}_k = -\frac{\partial H}{\partial k} = \left[-\lambda_a\frac{\partial M_{1i}}{\partial k}u_i - \lambda_h\frac{\partial M_{2i}}{\partial k}u_i - \lambda_k\frac{\partial M_{3i}}{\partial k}u_i - \lambda_p\frac{\partial M_{4i}}{\partial k}u_i - \lambda_q\frac{\partial M_{5i}}{\partial k}u_i - \lambda_F\frac{\partial M_{6i}^F}{\partial k}u_i\right]f_t - \lambda_F\frac{\partial}{\partial k}\left(\frac{na}{r}\right) \tag{27}$$

$$\dot{\lambda}_p = -\frac{\partial H}{\partial p} = \left[-\lambda_a\frac{\partial M_{1i}}{\partial p}u_i - \lambda_h\frac{\partial M_{2i}}{\partial p}u_i - \lambda_k\frac{\partial M_{3i}}{\partial p}u_i - \lambda_p\frac{\partial M_{4i}}{\partial p}u_i - \lambda_q\frac{\partial M_{5i}}{\partial p}u_i - \lambda_F\frac{\partial M_{6i}^F}{\partial p}u_i\right]f_t \tag{28}$$

$$\dot{\lambda}_q = -\frac{\partial H}{\partial q} = \left[-\lambda_a\frac{\partial M_{1i}}{\partial q}u_i - \lambda_h\frac{\partial M_{2i}}{\partial q}u_i - \lambda_k\frac{\partial M_{3i}}{\partial q}u_i - \lambda_p\frac{\partial M_{4i}}{\partial q}u_i - \lambda_q\frac{\partial M_{5i}}{\partial q}u_i - \lambda_F\frac{\partial M_{6i}^F}{\partial q}u_i\right]f_t \tag{29}$$

$$\dot{\lambda}_F = -\frac{\partial H}{\partial F} = \left[-\lambda_a\frac{\partial M_{1i}}{\partial F}u_i - \lambda_h\frac{\partial M_{2i}}{\partial F}u_i - \lambda_k\frac{\partial M_{3i}}{\partial F}u_i - \lambda_p\frac{\partial M_{4i}}{\partial F}u_i - \lambda_q\frac{\partial M_{5i}}{\partial F}u_i - \lambda_F\frac{\partial M_{6i}^F}{\partial F}u_i\right]f_t - \lambda_F\frac{\partial}{\partial F}\left(\frac{na}{r}\right) \tag{30}$$

However, the $\partial M/\partial z$ partials must be derived with the assumption that $\partial F/\partial h = \partial F/\partial k = \partial F/\partial a = 0$, consistent with assuming F as an independent variable. For the first five rows of the M matrix, these partials were derived[6] and are included in Appendix A for completeness. This formulation is the only one that results in mixed second-order partials for both X_1 and Y_1, namely, $\partial^2 X_1/\partial h\partial k \neq \partial^2 X_1/\partial k\partial h$ and $\partial^2 Y_1/\partial h\partial k \neq \partial^2 Y_1/\partial k\partial h$ not being identical as can be seen from Eqs. (A35), (A36), (A39), and (A40). This lack of symmetry is due to the independence of F with respect to both h and k. This dependence can be restored if the Hamiltonian in Eq. (23) is written without the $\lambda_F na/r$ term such that Eq. (6) is written as $\dot{F}' = (\partial F/\partial \dot{r})^T \cdot \hat{u} f_t$ with F' representing the sixth orbital element that remains constant in the absence of any thrust perturbation. The eccentric longitude F will then be obtained by a separate integration such that $\dot{F} = \dot{F}' + na/r$. When only the first five elements are considered, the Euler–Lagrange equations can be written either as in Ref. 7 or as developed in this paper, the results being identical when averaging is used because then the dynamic and adjoint equations are effectively decoupled from the orbital position.

However, when the Hamiltonian is written as in Eq. (23), we must accept that F is independent of a, h, and k when deriving the adjoint equations. There is no ambiguity when the full six differential equations, mainly Eqs. (1–6), are used as in the present paper within the context of precision integration. The mixed second-order partials of X_1 and Y_1 are not identical even though X_1 and Y_1 are continuous in h and k. Although X_1 and Y_1 are continuous in these two variables, the first-order partials $\partial X_1/\partial h$, $\partial X_1/\partial k$, $\partial Y_1/\partial h$, and $\partial Y_1/\partial k$ appearing explicitly in the elements of the M matrix and giving rise to the second-order partials in $\partial M/\partial z$ are combined with other terms, as it is clear from Eqs. (12), (13), (16), and (17), such that the totality of the $\hat{f}$ and $\hat{g}$ components of the pertinent elements of M must be considered as unique functions of the elements after replacement of the $\partial X_1/\partial h$, $\partial X_1/\partial k$, $\partial Y_1/\partial h$, and $\partial Y_1/\partial k$ partials by their functional representation given in Eqs. (18–21). We should therefore revisit the assumptions made to this regard in Ref. 6 and not worry about the explicit presence of these first-order partials in M because Eqs. (12), (13), (16), and (17) are shown as they are for convenience and compactness in writing. During the generation of the $\partial M/\partial z$ partials in the Appendices, care must be taken in writing the partials of X_1 and Y_1 with respect to h and k whenever X_1 and Y_1 appear explicitly in the matrix M. As stated earlier, these partials must not allow for the variation of F with respect to h and k such that

$$\left(\frac{\partial X_1}{\partial h}\right)_F = a\left[-(hc_F - ks_F)\left(\beta + \frac{h^2\beta^3}{1-\beta}\right) - h\beta c_F\right]$$

$$\left(\frac{\partial X_1}{\partial k}\right)_F = -a\left[(hc_F - ks_F)\frac{hk\beta^3}{1-\beta} + 1 - h\beta s_F\right]$$

$$\left(\frac{\partial Y_1}{\partial h}\right)_F = a\left[(hc_F - ks_F)\frac{hk\beta^3}{1-\beta} - 1 + k\beta c_F\right]$$

$$\left(\frac{\partial Y_1}{\partial k}\right)_F = a\left[(hc_F - ks_F)\left(\beta + \frac{k^2\beta^3}{1-\beta}\right) - k\beta s_F\right]$$

This distinction between $\partial X_1/\partial h$ and $(\partial X_1/\partial h)_F$ as well as the other three pairs of partials was not properly documented in the original conference paper in Ref. 13. An error in Eq. (131) of Ref. 13 in the form of h instead of k that appears in the second term has also been corrected in the corresponding Eq. (B36) of Appendix B of the present paper. The numerical results presented in this paper and based on the exact six-state dynamics validate these assertions. For the sixth row of the M matrix, the nonzero partials are given in Appendix B.

Numerical Example

Let us now try to duplicate the transfer trajectory used in Ref. 12. The minimum-time transfer solution must satisfy the transversality condition at the unknown final time $H_f = 0$ for the Hamiltonian $H = -1 + \boldsymbol{\lambda}_z^T \dot{z}$ or equivalently $H_f = 1$ for our Hamiltonian $H = \boldsymbol{\lambda}_z^T \dot{z}$. The initial values of the five Lagrange multipliers $(\lambda_a)_0$, $(\lambda_h)_0$, $(\lambda_k)_0$, $(\lambda_p)_0$, and $(\lambda_q)_0$ are guessed as well as the total transfer time and the initial value of the eccentric longitude, and the shooting method is used to integrate from the initial given state with $(\lambda_F)_0 = 0$ in an iterative manner until the required final state is matched. In practice, the following objective function is minimized:

$$F_{\text{obj}} = w_1(a_f - a_T)^2 + w_2(h_f - h_T)^2 + w_3(k_f - k_T)^2 + w_4(p_f - p_T)^2 + w_5(q_f - q_T)^2 + w_6[(\lambda_F)_f - 0]^2 + w_7(H_f - 1)^2$$

where the w_j are weights associated to each element and the f subscript represents the values of the equinoctial elements, the multiplier λ_F, and the Hamiltonian H at the current final time of that particular iteration. The subscript T represents the targeted values that the iterator must match closely for a converged solution.

Using the F superscript for the present F formulation and the λ superscript for the λ formulation of Ref. 7, a canonical transformation between the set (a, h, k, p, q, λ) and the set (a, h, k, p, q, F) will require that the following condition hold true:

$$\lambda_a^\lambda \, da + \lambda_h^\lambda \, dh + \lambda_k^\lambda \, dk + \lambda_p^\lambda \, dp + \lambda_q^\lambda \, dq + \lambda_\lambda^\lambda \, d\lambda + H^\lambda \, dt = \lambda_a^F \, da + \lambda_h^F \, dh + \lambda_k^F \, dk + \lambda_p^F \, dp + \lambda_q^F \, dq + \lambda_F^F \, dF + H^F \, dt \tag{31}$$

Because the dynamic system is autonomous in both formulations, $H^\lambda = H^F = 1$ throughout the transfer, and from Kepler's equation $\lambda = F - ks_F + hc_F$, we have $d\lambda = (1 - kc_F - hs_F)\, dF - s_F \, dk + c_F \, dh$, or

$$d\lambda = (r/a)\, dF - s_F \, dk + c_F \, dh \tag{32}$$

which when used in Eq. (31) yields the following six relationships:

$$\lambda_a^F = \lambda_a^\lambda \tag{33}$$

$$\lambda_h^F = \lambda_h^\lambda + \lambda_\lambda^\lambda c_F \tag{34}$$

$$\lambda_k^F = \lambda_k^\lambda - \lambda_\lambda^\lambda s_F \tag{35}$$

$$\lambda_p^F = \lambda_p^\lambda \tag{36}$$

$$\lambda_q^F = \lambda_q^\lambda \tag{37}$$

$$\lambda_F^F = (r/a)\lambda_\lambda^\lambda \tag{38}$$

Both initial and final eccentric longitudes are free, and therefore they are optimized because $(\lambda_F)_0 = 0$ is imposed, and $(\lambda_F)_f = 0$ is iterated on. We can also verify that the derivations presented in this paper are indeed correct and error free by running an open-loop trajectory using the solution shown in Ref. 12. Using $a_0 = 7000$ km, $e_0 = 0$, $i_0 = 28.5$ deg, $\Omega_0 = 0$ deg, and $\omega_0 = 0$ deg, the target conditions $a_f = 42{,}000$ km, $e_f = 10^{-3}$, $i_f = 1$ deg, and $\Omega_f = \omega_f = 0$ deg, and the acceleration $f_t = 9.8 \times 10^{-5}$ km/s^2, the λ formulation resulted in the following solution[12] given by $(\lambda_a)_0 = 4.675229762$ s/km, $(\lambda_h)_0 = 5.413413947 \times 10^2$ s, $(\lambda_k)_0 = -9.202702084 \times 10^3$ s, $(\lambda_p)_0 = 1.778011878 \times 10^1$ s, and $(\lambda_q)_0 = -2.258455855 \times 10^4$ s, with an optimized initial mean longitude $(\lambda)_0 = -2.274742851$ rad corresponding to an initial mean anomaly $M_0 = -130.3331648$ deg, and the minimum time of $t_f = 58089.90058$ s. Because $(\lambda_\lambda)_0 = 0$ was used at time zero to start the integration, and because $(r)_0 = a_0$ due to the initial circular condition, Eqs. (33–38) show that we can select λ_a, λ_h, λ_k, λ_p, λ_q, and $\lambda_F = 0$ at the initial time to be identical to the values shown earlier. Furthermore, $(F)_0 = (\lambda)_0 = (M)_0$ because $e_0 = 0$ and $\Omega_0 = \omega_0 = 0$ deg in this example. The open-loop integration to t_f results in the final parameters $a_f = 41{,}999.991$ km, $e_f = 9.9833 \times 10^{-4}$, $i_f = 0.999797$ deg, $\Omega_f = 3.435 \times 10^{-4}$ deg, $\omega_f = 359.992909$ deg, and $F_f = 46.206027$ deg with $(\lambda_F)_f = 9.089 \times 10^{-3}$ s/rad duplicating the minimum-time transfer example of Ref. 12. This F_f value corresponds to $M_f = 46.171481$ deg, which is close to $M_f = 46.146408$ deg obtained in Ref. 12. Figure 1 shows how F and M vary during the short optimal flight starting from the same initial value of 229.666835 deg or -130.333164 deg, with the eccentric longitude cycling between 0 and 360 deg. Figures 2–7 show the evolution of the various multipliers and compare them to the corresponding multipliers of the λ formulation of Ref. 12 with identical λ_a, λ_p, and λ_q as predicted by Eqs. (33), (36), and (37). Both λ_λ and λ_F start and end at zero as required for the solution of the free-free transfer because both initial and final

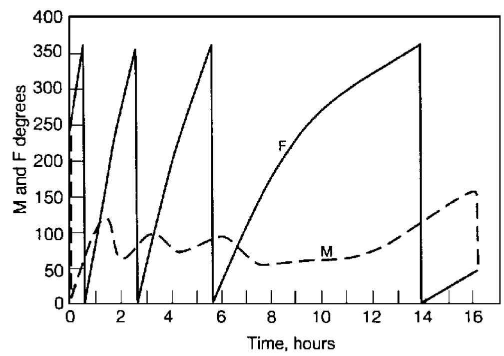

Fig. 1 Mean anomaly and eccentric longitude time histories during optimal transfer.

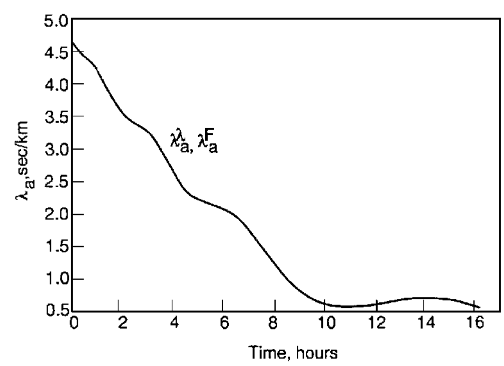

Fig. 2 Evolution of λ_a^λ and λ_a^F multipliers during optimal transfer.

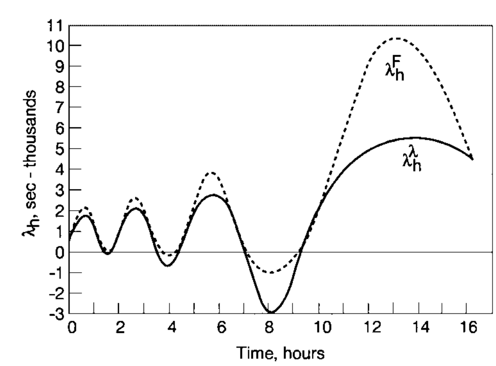

Fig. 3 Evolution of λ_h^λ and λ_h^F multipliers during optimal transfer.

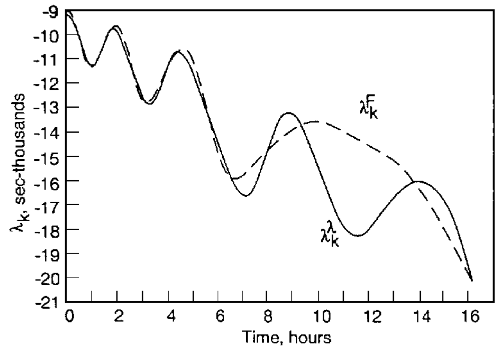

Fig. 4 Evolution of λ_k^λ and λ_k^F multipliers during optimal transfer.

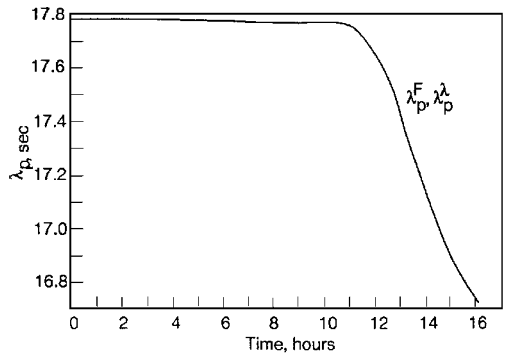

Fig. 5 Evolution of λ_p^λ and λ_p^F multipliers during optimal transfer.

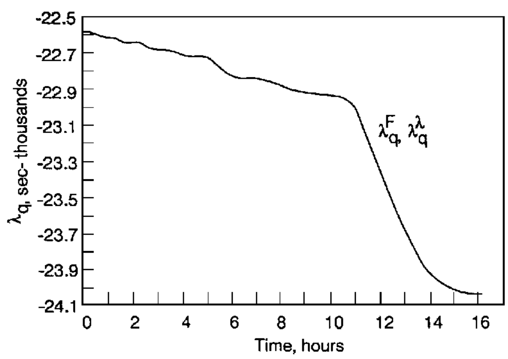

Fig. 6 Evolution of λ_q^λ and λ_q^F multipliers during optimal transfer.

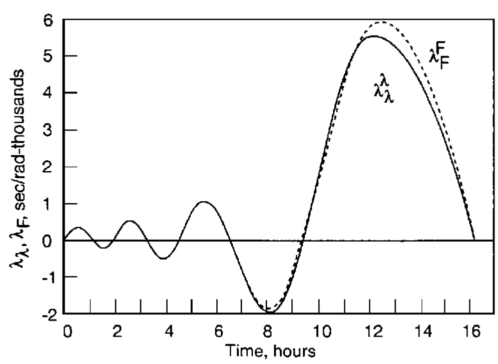

Fig. 7 Evolution of λ_λ^λ and λ_F^F multipliers during optimal transfer.

longitudes are free and are therefore optimized to yield the absolute minimum-time transfer. Finally, Figs. 8 and 9 depict the variations of the Earth-centered inertial Cartesian position components with Fig. 9 showing clearly how the orbit initially inclined at 28.5 deg rotates near the end of its run to match the 1 deg target. These expressions clearly show why the in-plane adjoints exhibit different plots with the exception of the various λ_a. Because our example transfer remains near circular throughout, r/a stays close to the unit value such that the λ_F and λ_λ curves are nearly identical. This near identity of the λ_F and λ_λ curves will not appear when the orbits are

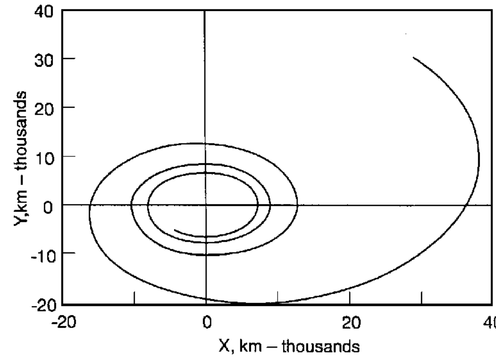

Fig. 8 Inertial Cartesian coordinates Y vs X during optimal transfer.

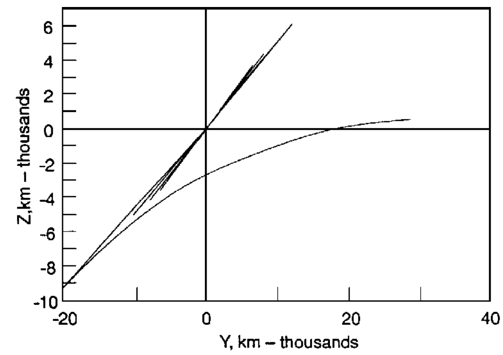

Fig. 9 Inertial Cartesian coordinates Z vs Y during optimal transfer.

elliptical because then r/a will effectively modulate the λ_λ curve to generate the λ_F curve.

Concluding Remarks

The mathematics of trajectory optimization for low-thrust orbit transfer and rendezvous are derived for the particular case where the eccentric longitude represents the sixth state variable. The use of this nonsingular set removes the need for solving the Kepler transcendental equation at each integration step because the eccentric longitude is now directly integrated. An example of minimum-time transfer using continuous constant acceleration obtained with a different formulation where the mean longitude represents the sixth state variable instead is duplicated by way of this formulation. The correspondence between the adjoints to the six elements of both formulations is also established, and the result is used to set the initial values of the six Lagrange multipliers corresponding to the present formulation to duplicate an open-loop transfer trajectory for validation and verification of the various mathematical derivations.

Appendix A: Nonzero Partials of the First Five Rows of Matrix M

$$\frac{\partial M_{11}}{\partial h} = \frac{2}{n^2 a}\frac{\partial \dot{X}_1}{\partial h} \tag{A1}$$

$$\frac{\partial M_{12}}{\partial h} = \frac{2}{n^2 a}\frac{\partial \dot{Y}_1}{\partial h} \tag{A2}$$

$$\frac{\partial M_{21}}{\partial h} = \frac{-h}{na^2 G}\left(\frac{\partial X_1}{\partial k} - \frac{h\beta}{n}\dot{X}_1\right) + \frac{G}{na^2}\left[\frac{\partial^2 X_1}{\partial h \partial k} - \frac{\dot{X}_1}{n}\left(\beta + \frac{h^2\beta^3}{1-\beta}\right) - \frac{h\beta}{n}\frac{\partial \dot{X}_1}{\partial h}\right] \tag{A3}$$

In a similar way

$$\frac{\partial M_{22}}{\partial h} = \frac{-h}{na^2 G}\left(\frac{\partial Y_1}{\partial k} - \frac{h\beta}{n}\dot{Y}_1\right) + \frac{G}{na^2}\left[\frac{\partial^2 Y_1}{\partial h \partial k} - \frac{\dot{Y}_1}{n}\left(\beta + \frac{h^2\beta^3}{1-\beta}\right) - \frac{h\beta}{n}\frac{\partial \dot{Y}_1}{\partial h}\right] \tag{A4}$$

$$\frac{\partial M_{23}}{\partial h} = \frac{hkG^{-3}}{na^2}(qY_1 - pX_1) + \frac{k[q(\partial Y_1/\partial h)_F - p(\partial X_1/\partial h)_F]}{na^2G} \tag{A5}$$

$$\frac{\partial M_{31}}{\partial h} = \frac{h}{na^2G}\left(\frac{\partial X_1}{\partial h} + k\beta\frac{\dot{X}_1}{n}\right) - \frac{G}{na^2}\left[\frac{\partial^2 X_1}{\partial h^2} + \frac{hk\beta^3}{1-\beta}\frac{\dot{X}_1}{n} + \frac{k\beta}{n}\frac{\partial \dot{X}_1}{\partial h}\right] \tag{A6}$$

$$\frac{\partial M_{32}}{\partial h} = \frac{h}{na^2G}\left(\frac{\partial Y_1}{\partial h} + k\beta\frac{\dot{Y}_1}{n}\right) - \frac{G}{na^2}\left[\frac{\partial^2 Y_1}{\partial h^2} + \frac{hk\beta^3}{1-\beta}\frac{\dot{Y}_1}{n} + \frac{k\beta}{n}\frac{\partial \dot{Y}_1}{\partial h}\right] \tag{A7}$$

$$\frac{\partial M_{33}}{\partial h} = \frac{-1}{na^2G}\left\{(qY_1 - pX_1) + h\left[q\left(\frac{\partial Y_1}{\partial h}\right)_F - p\left(\frac{\partial X_1}{\partial h}\right)_F\right]\right\} - \frac{h^2(qY_1 - pX_1)}{na^2G^3} \tag{A8}$$

$$\frac{\partial M_{43}}{\partial h} = \frac{K}{2na^2G}\left[\left(\frac{\partial Y_1}{\partial h}\right)_F + \frac{hY_1}{G^2}\right] \tag{A9}$$

$$\frac{\partial M_{53}}{\partial h} = \frac{K}{2na^2G}\left[\left(\frac{\partial X_1}{\partial h}\right)_F + \frac{hX_1}{G^2}\right] \tag{A10}$$

$$\frac{\partial M_{11}}{\partial k} = \frac{2}{n^2a}\frac{\partial \dot{X}_1}{\partial k} \tag{A11}$$

$$\frac{\partial M_{12}}{\partial k} = \frac{2}{n^2a}\frac{\partial \dot{Y}_1}{\partial k} \tag{A12}$$

$$\frac{\partial M_{21}}{\partial k} = \frac{-k}{na^2G}\left(\frac{\partial X_1}{\partial k} - \frac{h\beta}{n}\dot{X}_1\right) + \frac{G}{na^2}\left[\frac{\partial^2 X_1}{\partial k^2} - \frac{hk\beta^3}{n(1-\beta)}\dot{X}_1 - \frac{h\beta}{n}\frac{\partial \dot{X}_1}{\partial k}\right] \tag{A13}$$

$$\frac{\partial M_{22}}{\partial k} = \frac{-k}{na^2G}\left(\frac{\partial Y_1}{\partial k} - \frac{h\beta}{n}\dot{Y}_1\right) + \frac{G}{na^2}\left[\frac{\partial^2 Y_1}{\partial k^2} - \frac{hk\beta^3}{n(1-\beta)}\dot{Y}_1 - \frac{h\beta}{n}\frac{\partial \dot{Y}_1}{\partial k}\right] \tag{A14}$$

$$\frac{\partial M_{23}}{\partial k} = \frac{(qY_1 - pX_1)}{na^2G} + \frac{1}{na^2G}\left\{k\left[q\left(\frac{\partial Y_1}{\partial k}\right)_F - p\left(\frac{\partial X_1}{\partial k}\right)_F\right] + \frac{k^2(qY_1 - pX_1)}{G^2}\right\} \tag{A15}$$

$$\frac{\partial M_{31}}{\partial k} = \frac{k}{na^2G}\left(\frac{\partial X_1}{\partial h} + k\beta\frac{\dot{X}_1}{n}\right) - \frac{G}{na^2}\left[\frac{\partial^2 X_1}{\partial k\partial h} + \left(\beta + \frac{k^2\beta^3}{1-\beta}\right)\frac{\dot{X}_1}{n} + \frac{k\beta}{n}\frac{\partial \dot{X}_1}{\partial k}\right] \tag{A16}$$

$$\frac{\partial M_{32}}{\partial k} = \frac{k}{na^2G}\left(\frac{\partial Y_1}{\partial h} + k\beta\frac{\dot{Y}_1}{n}\right) - \frac{G}{na^2}\left[\frac{\partial^2 Y_1}{\partial k\partial h} + \left(\beta + \frac{k^2\beta^3}{1-\beta}\right)\frac{\dot{Y}_1}{n} + \frac{k\beta}{n}\frac{\partial \dot{Y}_1}{\partial k}\right] \tag{A17}$$

$$\frac{\partial M_{33}}{\partial k} = \frac{-h}{na^2G}\left[q\left(\frac{\partial Y_1}{\partial k}\right)_F - p\left(\frac{\partial X_1}{\partial k}\right)_F\right] - \frac{hk}{na^2G^3}(qY_1 - pX_1) \tag{A18}$$

$$\frac{\partial M_{43}}{\partial k} = \frac{K}{2na^2G}\left(\frac{\partial Y_1}{\partial k}\right)_F + \frac{kK}{2na^2G^3}Y_1 \tag{A19}$$

$$\frac{\partial M_{53}}{\partial k} = \frac{K}{2na^2G}\left(\frac{\partial X_1}{\partial k}\right)_F + \frac{kK}{2na^2G^3}X_1 \tag{A20}$$

The only nonzero partials with respect to p are

$$\frac{\partial M_{23}}{\partial p} = \frac{-kX_1}{na^2G} \tag{A21}$$

$$\frac{\partial M_{33}}{\partial p} = \frac{hX_1}{na^2G} \tag{A22}$$

$$\frac{\partial M_{43}}{\partial p} = \frac{pY_1}{na^2G} \tag{A23}$$

$$\frac{\partial M_{53}}{\partial p} = \frac{pX_1}{na^2G} \tag{A24}$$

The only nonzero partials with respect to q are

$$\frac{\partial M_{23}}{\partial q} = \frac{kY_1}{na^2G} \tag{A25}$$

$$\frac{\partial M_{33}}{\partial q} = \frac{-hY_1}{na^2G} \tag{A26}$$

$$\frac{\partial M_{43}}{\partial q} = \frac{qY_1}{na^2G} \tag{A27}$$

$$\frac{\partial M_{53}}{\partial q} = \frac{qX_1}{na^2G} \tag{A28}$$

The accessory partials appearing in the definition of the preceding partial derivatives are obtained from Eqs. (9), (10), and (18–21) by assuming that $\partial F/\partial h = \partial F/\partial k = 0$:

$$\frac{\partial \dot{X}_1}{\partial h} = \frac{a}{r}\dot{X}_1 s_F + \frac{na^2}{r}\left[h\beta s_F + (kc_F + hs_F)\left(\beta + \frac{h^2\beta^3}{1-\beta}\right)\right] \tag{A29}$$

$$\frac{\partial \dot{X}_1}{\partial k} = \frac{a}{r}\dot{X}_1 c_F + \frac{na^2}{r}\left[h\beta c_F + (kc_F + hs_F)\frac{hk\beta^3}{1-\beta}\right] \tag{A30}$$

$$\frac{\partial \dot{Y}_1}{\partial h} = \frac{a}{r}\dot{Y}_1 s_F + \frac{na^2}{r}\left[-k\beta s_F - (kc_F + hs_F)\frac{hk\beta^3}{1-\beta}\right] \tag{A31}$$

$$\frac{\partial \dot{Y}_1}{\partial k} = \frac{a}{r}\dot{Y}_1 c_F + \frac{na^2}{r}\left[-k\beta c_F - (kc_F + hs_F)\left(\beta + \frac{k^2\beta^3}{1-\beta}\right)\right] \tag{A32}$$

$$\frac{\partial^2 X_1}{\partial h^2} = a\left\{-\left(1+\frac{a}{r}\right)c_F\left(\beta+\frac{h^2\beta^3}{1-\beta}\right) - \frac{h\beta^3}{1-\beta}(hc_F - ks_F)\right.$$
$$\left.\times\left[3+\frac{h^2\beta^2(3-2\beta)}{(1-\beta)^2}\right] - \frac{a^2}{r^2}s_F c_F(h\beta - s_F)\right\} \tag{A33}$$

$$\frac{\partial^2 X_1}{\partial k^2} = -a\left\{-s_F\left(1+\frac{a}{r}\right)\frac{hk\beta^3}{1-\beta} + \frac{h\beta^3}{1-\beta}(hc_F - ks_F)\right.$$
$$\left.\times\left[1+\frac{k^2\beta^2(3-2\beta)}{(1-\beta)^2}\right] - \frac{a^2}{r^2}s_F c_F(h\beta - s_F)\right\} \tag{A34}$$

$$\frac{\partial^2 X_1}{\partial h \partial k} = -a\left\{c_F\frac{hk\beta^3}{1-\beta} + \frac{k\beta^3}{1-\beta}(hc_F - ks_F)\right.$$
$$\times\left[1+\frac{h^2\beta^2(3-2\beta)}{(1-\beta)^2}\right] + \frac{a^2}{r^2}s_F^2(s_F - h\beta)$$
$$\left.-\frac{a}{r}s_F\left(\beta+\frac{h^2\beta^3}{1-\beta}\right)\right\} \tag{A35}$$

$$\frac{\partial^2 X_1}{\partial k \partial h} = a\left\{s_F\left(\beta+\frac{h^2\beta^3}{1-\beta}\right) - \frac{k\beta^3}{1-\beta}(hc_F - ks_F)\right.$$
$$\left.\times\left[1+\frac{h^2\beta^2(3-2\beta)}{(1-\beta)^2}\right] - \frac{a^2}{r^2}c_F^2(h\beta - s_F) - \frac{a}{r}c_F\frac{hk\beta^3}{1-\beta}\right\} \tag{A36}$$

$$\frac{\partial^2 Y_1}{\partial h^2} = a\left\{c_F\left(1+\frac{a}{r}\right)\frac{hk\beta^3}{1-\beta} + \frac{k\beta^3}{1-\beta}(hc_F - ks_F)\right.$$
$$\left.\times\left[1+\frac{h^2\beta^2(3-2\beta)}{(1-\beta)^2}\right] + \frac{a^2}{r^2}s_F c_F(k\beta - c_F)\right\} \tag{A37}$$

$$\frac{\partial^2 Y_1}{\partial k^2} = a\left\{-s_F\left(1+\frac{a}{r}\right)\left(\beta+\frac{k^2\beta^3}{1-\beta}\right)\right.$$
$$+\frac{k\beta^3}{1-\beta}(hc_F - ks_F)\left[3+\frac{k^2\beta^2(3-2\beta)}{(1-\beta)^2}\right]$$
$$\left.+\frac{a^2}{r^2}s_F c_F(c_F - k\beta)\right\} \tag{A38}$$

$$\frac{\partial^2 Y_1}{\partial h \partial k} = a\left\{c_F\left(\beta+\frac{k^2\beta^3}{1-\beta}\right) + \frac{h\beta^3}{1-\beta}(hc_F - ks_F)\right.$$
$$\left.\times\left[1+\frac{k^2\beta^2(3-2\beta)}{(1-\beta)^2}\right] + \frac{a^2}{r^2}s_F^2(c_F - k\beta) - \frac{a}{r}s_F\frac{hk\beta^3}{1-\beta}\right\} \tag{A39}$$

$$\frac{\partial^2 Y_1}{\partial k \partial h} = a\left\{-s_F\frac{hk\beta^3}{1-\beta} + \frac{h\beta^3}{1-\beta}(hc_F - ks_F)\right.$$
$$\times\left[1+\frac{k^2\beta^2(3-2\beta)}{(1-\beta)^2}\right] + \frac{a^2}{r^2}c_F^2(k\beta - c_F)$$
$$\left.+\frac{a}{r}c_F\left(\beta+\frac{k^2\beta^3}{1-\beta}\right)\right\} \tag{A40}$$

$$\frac{\partial^2 X_1}{\partial a \partial h} = \frac{1}{a}\frac{\partial X_1}{\partial h} \tag{A41}$$

$$\frac{\partial^2 X_1}{\partial a \partial k} = \frac{1}{a}\frac{\partial X_1}{\partial k} \tag{A42}$$

$$\frac{\partial^2 Y_1}{\partial a \partial h} = \frac{1}{a}\frac{\partial Y_1}{\partial h} \tag{A43}$$

$$\frac{\partial^2 Y_1}{\partial a \partial k} = \frac{1}{a}\frac{\partial Y_1}{\partial k} \tag{A44}$$

where the $\partial X_1/\partial h$, $\partial X_1/\partial k$, $\partial Y_1/\partial h$, and $\partial Y_1/\partial k$ partials just appearing are given by Eqs. (18–21). The last four second-order partials are needed to define the $\partial M/\partial a$ partials in Eq. (A49), produced by assuming $\partial F/\partial a = 0$. We have

$$\frac{\partial X_1}{\partial a} = \frac{X_1}{a} \tag{A45}$$

$$\frac{\partial Y_1}{\partial a} = \frac{Y_1}{a} \tag{A46}$$

$$\frac{\partial \dot{X}_1}{\partial a} = -\frac{na}{2r}\left[hk\beta c_F - (1-h^2\beta)s_F\right] = -\frac{\dot{X}_1}{2a} \tag{A47}$$

$$\frac{\partial \dot{Y}_1}{\partial a} = -\frac{na}{2r}\left[hk\beta s_F - (1-k^2\beta)c_F\right] = -\frac{\dot{Y}_1}{2a} \tag{A48}$$

This, together with the second-order partials of Eqs. (A41–A44) give

$$\frac{\partial M}{\partial a} = \frac{1}{2a}\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 \\ & & 1 & 0 & 0 \\ & & & 1 & 0 \\ 0 & & & & 1 \end{bmatrix} M \tag{A49}$$

Appendix B: Partials of the Sixth Row of Matrix *M*

$$\frac{\partial M_{61}^F}{\partial a} = \frac{a}{r}\left[\frac{\partial M_{61}}{\partial a} + s_F\frac{\partial M_{31}}{\partial a} - c_F\frac{\partial M_{21}}{\partial a}\right] \tag{B1}$$

$$\frac{\partial M_{62}^F}{\partial a} = \frac{a}{r}\left[\frac{\partial M_{62}}{\partial a} + s_F\frac{\partial M_{32}}{\partial a} - c_F\frac{\partial M_{22}}{\partial a}\right] \tag{B2}$$

$$\frac{\partial M_{63}^F}{\partial a} = \frac{a}{r}\left[\frac{\partial M_{63}}{\partial a} + s_F\frac{\partial M_{33}}{\partial a} - c_F\frac{\partial M_{23}}{\partial a}\right] \tag{B3}$$

$$\frac{\partial}{\partial a}\left(\frac{na}{r}\right) = -\frac{3}{2}\frac{\mu^{\frac{1}{2}}a^{-\frac{3}{2}}}{r} \tag{B4}$$

$$\frac{\partial M_{61}^F}{\partial h} = \frac{a^2}{r^2}s_F(M_{61} + s_F M_{31} - c_F M_{21})$$
$$+\frac{a}{r}\left[\frac{\partial M_{61}}{\partial h} + s_F\frac{\partial M_{31}}{\partial h} - c_F\frac{\partial M_{21}}{\partial h}\right] \tag{B5}$$

$$\frac{\partial M_{62}^F}{\partial h} = \frac{a^2}{r^2}s_F(M_{62} + s_F M_{32} - c_F M_{22})$$
$$+\frac{a}{r}\left[\frac{\partial M_{62}}{\partial h} + s_F\frac{\partial M_{32}}{\partial h} - c_F\frac{\partial M_{22}}{\partial h}\right] \tag{B6}$$

$$\frac{\partial M_{63}^F}{\partial h} = \frac{a^2}{r^2}s_F(M_{63} + s_F M_{33} - c_F M_{23})$$
$$+\frac{a}{r}\left[\frac{\partial M_{63}}{\partial h} + s_F\frac{\partial M_{33}}{\partial h} - c_F\frac{\partial M_{23}}{\partial h}\right] \tag{B7}$$

$$\frac{\partial}{\partial h}\left(\frac{na}{r}\right) = \frac{na^2}{r^2}s_F \tag{B8}$$

$$\frac{\partial M_{61}^F}{\partial k} = \frac{a^2}{r^2}c_F(M_{61} + s_F M_{31} - c_F M_{21}) + \frac{a}{r}\left[\frac{\partial M_{61}}{\partial k} + s_F\frac{\partial M_{31}}{\partial k} - c_F\frac{\partial M_{21}}{\partial k}\right] \tag{B9}$$

$$\frac{\partial M_{62}^F}{\partial k} = \frac{a^2}{r^2}c_F(M_{62} + s_F M_{32} - c_F M_{22}) + \frac{a}{r}\left[\frac{\partial M_{62}}{\partial k} + s_F\frac{\partial M_{32}}{\partial k} - c_F\frac{\partial M_{22}}{\partial k}\right] \tag{B10}$$

$$\frac{\partial M_{63}^F}{\partial k} = \frac{a^2}{r^2}c_F(M_{63} + s_F M_{33} - c_F M_{23}) + \frac{a}{r}\left[\frac{\partial M_{63}}{\partial k} + s_F\frac{\partial M_{33}}{\partial k} - c_F\frac{\partial M_{23}}{\partial k}\right] \tag{B11}$$

$$\frac{\partial}{\partial k}\left(\frac{na}{r}\right) = \frac{na^2}{r^2}c_F \tag{B12}$$

$$\frac{\partial M_{63}^F}{\partial p} = \frac{a}{r}\left(\frac{\partial M_{63}}{\partial p} + s_F\frac{\partial M_{33}}{\partial p} - c_F\frac{\partial M_{23}}{\partial p}\right) \tag{B13}$$

$$\frac{\partial M_{63}^F}{\partial q} = \frac{a}{r}\left(\frac{\partial M_{63}}{\partial q} + s_F\frac{\partial M_{33}}{\partial q} - c_F\frac{\partial M_{23}}{\partial q}\right) \tag{B14}$$

$$\frac{\partial M_{61}^F}{\partial F} = -\frac{a^2}{r^2}(ks_F - hc_F)(M_{61} + s_F M_{31} - c_F M_{21}) + \frac{a}{r}\left[\frac{\partial M_{61}}{\partial F} + s_F\frac{\partial M_{31}}{\partial F} - c_F\frac{\partial M_{21}}{\partial F} + c_F M_{31} + s_F M_{21}\right] \tag{B15}$$

$$\frac{\partial M_{62}^F}{\partial F} = -\frac{a^2}{r^2}(ks_F - hc_F)(M_{62} + s_F M_{32} - c_F M_{22}) + \frac{a}{r}\left[\frac{\partial M_{62}}{\partial F} + s_F\frac{\partial M_{32}}{\partial F} - c_F\frac{\partial M_{22}}{\partial F} + c_F M_{32} + s_F M_{22}\right] \tag{B16}$$

$$\frac{\partial M_{63}^F}{\partial F} = -\frac{a^2}{r^2}(ks_F - hc_F)(M_{63} + s_F M_{33} - c_F M_{23}) + \frac{a}{r}\left[\frac{\partial M_{63}}{\partial F} + s_F\frac{\partial M_{33}}{\partial F} - c_F\frac{\partial M_{23}}{\partial F} + c_F M_{33} + s_F M_{23}\right] \tag{B17}$$

Finally, the partial derivatives of M with respect to F that are not identically equal to zero are given by

$$\frac{\partial M_{11}}{\partial F} = \frac{2}{n^2 a}\frac{\partial \dot{X}_1}{\partial F} \tag{B18}$$

$$\frac{\partial M_{12}}{\partial F} = \frac{2}{n^2 a}\frac{\partial \dot{Y}_1}{\partial F} \tag{B19}$$

$$\frac{\partial M_{21}}{\partial F} = \frac{G}{na^2}\left(\frac{\partial^2 X_1}{\partial F \partial k} - \frac{h\beta}{n}\frac{\partial \dot{X}_1}{\partial F}\right) \tag{B20}$$

$$\frac{\partial M_{22}}{\partial F} = \frac{G}{na^2}\left(\frac{\partial^2 Y_1}{\partial F \partial k} - \frac{h\beta}{n}\frac{\partial \dot{Y}_1}{\partial F}\right) \tag{B21}$$

$$\frac{\partial M_{23}}{\partial F} = \frac{k[q(\partial Y_1/\partial F) - p(\partial X_1/\partial F)]}{na^2 G} \tag{B22}$$

$$\frac{\partial M_{31}}{\partial F} = -\frac{G}{na^2}\left(\frac{\partial^2 X_1}{\partial F \partial h} + \frac{k\beta}{n}\frac{\partial \dot{X}_1}{\partial F}\right) \tag{B23}$$

$$\frac{\partial M_{32}}{\partial F} = -\frac{G}{na^2}\left(\frac{\partial^2 Y_1}{\partial F \partial h} + \frac{k\beta}{n}\frac{\partial \dot{Y}_1}{\partial F}\right) \tag{B24}$$

$$\frac{\partial M_{33}}{\partial F} = \frac{-h[q(\partial Y_1/\partial F) - p(\partial X_1/\partial F)]}{na^2 G} \tag{B25}$$

$$\frac{\partial M_{43}}{\partial F} = \frac{K}{2na^2 G}\frac{\partial Y_1}{\partial F} \tag{B26}$$

$$\frac{\partial M_{53}}{\partial F} = \frac{K}{2na^2 G}\frac{\partial X_1}{\partial F} \tag{B27}$$

$$\frac{\partial M_{61}}{\partial F} = \frac{1}{na^2}\left[-2\frac{\partial X_1}{\partial F} + G\left(h\beta\frac{\partial^2 X_1}{\partial F \partial h} + k\beta\frac{\partial^2 X_1}{\partial F \partial k}\right)\right] \tag{B28}$$

$$\frac{\partial M_{62}}{\partial F} = \frac{1}{na^2}\left[-2\frac{\partial Y_1}{\partial F} + G\left(h\beta\frac{\partial^2 Y_1}{\partial F \partial h} + k\beta\frac{\partial^2 Y_1}{\partial F \partial k}\right)\right] \tag{B29}$$

$$\frac{\partial M_{63}}{\partial F} = \frac{[q(\partial Y_1/\partial F) - p(\partial X_1/\partial F)]}{na^2 G} \tag{B30}$$

The auxiliary partials are

$$\frac{\partial X_1}{\partial F} = a\left[hk\beta c_F - (1 - h^2\beta)s_F\right] \tag{B31}$$

$$\frac{\partial Y_1}{\partial F} = a\left[-hk\beta s_F + (1 - k^2\beta)c_F\right] \tag{B32}$$

$$\frac{\partial \dot{X}_1}{\partial F} = -\frac{a}{r}(ks_F - hc_F)\dot{X}_1 + \frac{a^2 n}{r}\left[-hk\beta s_F - (1 - h^2\beta)c_F\right] \tag{B33}$$

$$\frac{\partial \dot{Y}_1}{\partial F} = -\frac{a}{r}(ks_F - hc_F)\dot{Y}_1 + \frac{a^2 n}{r}\left[-hk\beta c_F - (1 - k^2\beta)s_F\right] \tag{B34}$$

$$\frac{\partial^2 X_1}{\partial F \partial h} = a\left[(hs_F + kc_F)\left(\beta + \frac{h^2\beta^3}{1-\beta}\right) + \frac{a^2}{r^2}(h\beta - s_F)(s_F - h) + \frac{a}{r}c_F^2\right] \tag{B35}$$

$$\frac{\partial^2 X_1}{\partial F \partial k} = -a\left[-(hs_F + kc_F)\frac{hk\beta^3}{1-\beta} + \frac{a^2}{r^2}(s_F - h\beta)(c_F - k) + \frac{a}{r}s_F c_F\right] \tag{B36}$$

$$\frac{\partial^2 Y_1}{\partial F \partial h} = a\left[-(hs_F + kc_F)\frac{hk\beta^3}{1-\beta} - \frac{a^2}{r^2}(k\beta - c_F)(s_F - h) + \frac{a}{r}s_F c_F\right] \tag{B37}$$

$$\frac{\partial^2 Y_1}{\partial F \partial k} = a\left[-(hs_F + kc_F)\left(\beta + \frac{k^2\beta^3}{1-\beta}\right) + \frac{a^2}{r^2}(c_F - k\beta)(c_F - k) - \frac{a}{r}s_F^2\right] \quad \text{(B38)}$$

$$\frac{\partial}{\partial F}\left(\frac{na}{r}\right) = -n\frac{a^2}{r^2}(ks_F - hc_F) \quad \text{(B39)}$$

Finally,

$$\frac{\partial M_{61}}{\partial a} = -\frac{M_{61}}{2a} + \frac{1}{na^2}\left[-2\frac{\partial X_1}{\partial a} + G\left(h\beta\frac{\partial^2 X_1}{\partial a\partial h} + k\beta\frac{\partial^2 X_1}{\partial a\partial k}\right)\right] \quad \text{(B40)}$$

$$\frac{\partial M_{62}}{\partial a} = -\frac{M_{62}}{2a} + \frac{1}{na^2}\left[-2\frac{\partial Y_1}{\partial a} + G\left(h\beta\frac{\partial^2 Y_1}{\partial a\partial h} + k\beta\frac{\partial^2 Y_1}{\partial a\partial k}\right)\right] \quad \text{(B41)}$$

$$\frac{\partial M_{63}}{\partial a} = -\frac{M_{63}}{2a} + \frac{1}{na^2}\left[\left(q\frac{\partial Y_1}{\partial a} - p\frac{\partial X_1}{\partial a}\right)G^{-1}\right] \quad \text{(B42)}$$

$$\frac{\partial M_{61}}{\partial h} = \frac{1}{na^2}\left\{-2\left(\frac{\partial X_1}{\partial h}\right)_F - h\beta G^{-1}\left(h\frac{\partial X_1}{\partial h} + k\frac{\partial X_1}{\partial k}\right) + G\left[\left(\beta + \frac{h^2\beta^3}{1-\beta}\right)\frac{\partial X_1}{\partial h} + \frac{hk\beta^3}{1-\beta}\frac{\partial X_1}{\partial k} + \beta\left(h\frac{\partial^2 X_1}{\partial h^2} + k\frac{\partial^2 X_1}{\partial h\partial k}\right)\right]\right\} \quad \text{(B43)}$$

$$\frac{\partial M_{62}}{\partial h} = \frac{1}{na^2}\left\{-2\left(\frac{\partial Y_1}{\partial h}\right)_F - h\beta G^{-1}\left(h\frac{\partial Y_1}{\partial h} + k\frac{\partial Y_1}{\partial k}\right) + G\left[\left(\beta + \frac{h^2\beta^3}{1-\beta}\right)\frac{\partial Y_1}{\partial h} + \frac{hk\beta^3}{1-\beta}\frac{\partial Y_1}{\partial k} + \beta\left(h\frac{\partial^2 Y_1}{\partial h^2} + k\frac{\partial^2 Y_1}{\partial h\partial k}\right)\right]\right\} \quad \text{(B44)}$$

$$\frac{\partial M_{63}}{\partial h} = \frac{G^{-1}}{na^2}\left\{\left[q\left(\frac{\partial Y_1}{\partial h}\right)_F - p\left(\frac{\partial X_1}{\partial h}\right)_F\right] + hG^{-2}(qY_1 - pX_1)\right\} \quad \text{(B45)}$$

$$\frac{\partial M_{61}}{\partial k} = \frac{1}{na^2}\left\{-2\left(\frac{\partial X_1}{\partial k}\right)_F - k\beta G^{-1}\left(h\frac{\partial X_1}{\partial h} + k\frac{\partial X_1}{\partial k}\right) + G\left[\left(\beta + \frac{k^2\beta^3}{1-\beta}\right)\frac{\partial X_1}{\partial k} + \frac{hk\beta^3}{1-\beta}\frac{\partial X_1}{\partial h} + \beta\left(h\frac{\partial^2 X_1}{\partial k\partial h} + k\frac{\partial^2 X_1}{\partial k^2}\right)\right]\right\} \quad \text{(B46)}$$

$$\frac{\partial M_{62}}{\partial k} = \frac{1}{na^2}\left\{-2\left(\frac{\partial Y_1}{\partial k}\right)_F - k\beta G^{-1}\left(h\frac{\partial Y_1}{\partial h} + k\frac{\partial Y_1}{\partial k}\right) + G\left[\left(\beta + \frac{k^2\beta^3}{1-\beta}\right)\frac{\partial Y_1}{\partial k} + \frac{hk\beta^3}{1-\beta}\frac{\partial Y_1}{\partial h} + \beta\left(h\frac{\partial^2 Y_1}{\partial k\partial h} + k\frac{\partial^2 Y_1}{\partial k^2}\right)\right]\right\} \quad \text{(B47)}$$

$$\frac{\partial M_{63}}{\partial k} = \frac{G^{-1}}{na^2}\left\{\left[q\left(\frac{\partial Y_1}{\partial k}\right)_F - p\left(\frac{\partial X_1}{\partial k}\right)_F\right] + kG^{-2}(qY_1 - pX_1)\right\} \quad \text{(B48)}$$

$$\frac{\partial M_{63}}{\partial p} = \frac{-X_1}{na^2 G} \quad \text{(B49)}$$

$$\frac{\partial M_{63}}{\partial q} = \frac{Y_1}{na^2 G} \quad \text{(B50)}$$

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